

Non-Gaussian Spectra in the Cosmic Microwave Background

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We propose a set of new statistics which can be extracted out of the angular distribution of the Fourier transform of the temperature anisotropies in the small field limit. They quantify generic non-Gaussian structure and complement the power spectrum in characterizing the sampled distribution function of a data set.

Recent years have seen scattered attempts at quantifying non-Gaussianity in cosmological data sets. It has become clear that there are two main lines of attack. One can focus on speculated sources of non-Gaussianity and design statistics which can best discriminate between them and Gaussian counterparts. This has been the approach in, for example, the study of cosmic defects such as strings or texture. The other, less prejudiced, approach is to define a general framework with which one can quantify deviations from Gaussianity. The main example of this strategy has been the n -point formalism which has been applied in a variety of settings¹. The redundancy and inefficiency of such a method makes it pressing to look for viable alternatives.

This second approach is in general too ambitious. One must define restrictions to make any form of quantitative analysis tractable. One can do this by establishing requirements. We shall describe a formalism² which

- preserves information, i.e. given N independent pixels this will supply N quantities, one of which is the power spectrum
- is defined in Fourier space, and therefore tailored for high resolution, small field, interferometric measurements

In this setting it makes sense to consider the Fourier transform:

$$\frac{\Delta T(\mathbf{x})}{T} = \int \frac{d\mathbf{k}}{2\pi} a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (1)$$

A Gaussian probability distribution function of the complex $a(\mathbf{k}_i)$ in a ring of fixed $|k|$ is given by:

$$F[a(\mathbf{k}_i)] = \frac{1}{(2\pi\sigma^2)^{m_k}} \exp\left(-\frac{1}{2\sigma_k^2} \sum_{i=1}^{m_k} |a(\mathbf{k}_i)|^2\right) \quad (2)$$

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where we have $2m_k = f_{sky}(2k+1)$ independent modes (f_{sky} is the fraction of the sky covered). Defining $a(\mathbf{k}_i) = \rho_i e^{i\phi_i}$ we can work in terms of m_k moduli ρ_i and m_k phases ϕ_i . The $\{\rho_i\}$ may be seen as Cartesian coordinates which we transform into polar coordinates. These consist of a radius r plus $m_k - 1$ angles $\tilde{\theta}_i$ given by

$$\rho_i = r \cos \tilde{\theta}_i \prod_{j=0}^{i-1} \sin \tilde{\theta}_j \quad (3)$$

with $\sin \tilde{\theta}_0 = \cos \tilde{\theta}_{m_k} = 1$. In terms of these variables the radius is related to the angular power spectrum by $C(k) = r^2/(2m_k)$. In general the first $m_k - 2$ angles $\tilde{\theta}_i$ vary between 0 and π and the last angle varies between 0 and 2π . However because all ρ_i are positive all angles are in $(0, \pi/2)$. In order to define $\tilde{\theta}_i$ variables which are uniformly distributed in Gaussian theories one may finally perform the transformation on each $\tilde{\theta}_i$:

$$\theta_i = \sin^{N_k-2i}(\tilde{\theta}_i) \quad (4)$$

so that for Gaussian theories one has:

$$F(r, \theta_i, \phi_i) = \frac{r^{N_k-1} e^{-r^2/(2\sigma_k^2)}}{2^{m_k-1} (m_k-1)!} \times 1 \times \prod_{i=1}^{m_k} \frac{1}{2\pi} \quad (5)$$

The factorization chosen shows that all new variables are independent random variables for Gaussian theories. r has a $\chi_{N_k}^2$ distribution, the “shape” variables θ_i are uniformly distributed in $(0, 1)$, and the phases ϕ_i are uniformly distributed in $(0, 2\pi)$.

The variables θ_i define a non-Gaussian shape spectrum, the *ring spectrum*. They may be computed from ring moduli ρ_i simply by

$$\theta_i = \left(\frac{\rho_{i+1}^2 + \dots + \rho_{m_k}^2}{\rho_i^2 + \dots + \rho_{m_k}^2} \right)^{m_k-i} \quad (6)$$

They describe how shapeful the perturbations are. If the perturbations are stringy then the maximal moduli will be much larger than the minimal moduli. If the perturbations are circular, then all moduli will be roughly the same. This favours some combinations of angles, which are otherwise uniformly distributed. In general any shapeful picture defines a line on the ring spectrum θ_i . A non-Gaussian theory ought to define a set of probable smooth ring spectra peaking along a ridge of typical shapes.

We can now construct an invariant for each adjacent pair of rings, solely out of the moduli. If we order the ρ_i for each ring, we can identify the maximum

moduli. Each of these moduli will have a specific direction in Fourier space; let \mathbf{k}_{max} and \mathbf{k}'_{max} be the directions where the maximal moduli are achieved. The angle

$$\psi(k, k') = \frac{1}{\pi} \text{ang}(\mathbf{k}_{max}, \mathbf{k}'_{max}) \quad (7)$$

will then produce an inter-ring correlator for the moduli, the *inter-ring spectra*. This is uniformly distributed in Gaussian theories in $(-1, 1)$. It gives us information on how connected the distribution of power is between the different scales.

We have therefore defined a transformation from the original modes into a set of variables $\{r, \theta, \phi, \psi\}$. The non-Gaussian spectra thus defined have a particularly simple distribution for Gaussian theories. We shall call perturbations for which the phases are not uniformly distributed localized perturbations. This is because if perturbations are made up of lumps statistically distributed but with well defined positions then the phases will appear highly correlated. We shall call perturbations for which the ring spectra are not uniformly distributed shapeful perturbations. This distinction is interesting as it is in principle possible for fluctuations to be localized but shapeless, or more surprisingly, to be shapeful but not localized. Finally we shall call perturbations for which the inter-ring spectra are not uniformly distributed, connected perturbations. This turns out to be one of the key features of perturbations induced by cosmic strings. These three definitions allow us to consider structure in various layers. White noise is the most structureless type of perturbation. Gaussian fluctuations allow for modulation, that is a non trivial power spectrum $C(k)$, but their structure stops there. Shape, localization, and connectedness constitute the three next levels of structure one might add on. Standard visual structure is contained within these definitions, but they allow for more abstract levels of structure.

References

1. P.J.E. Peebles, *The Large Scale Structure of the Universe*, Princeton University Press, (1980)
2. P. G. Ferreira, J. Magueijo, *Phys. Rev. D*, accepted, (1997)